ON A PROBLEM IN TRACKING

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We consider the problem of bringing the controlled motion $z(t)$ into a neighborhood of the random point $y(t)$. The displacements of $y(t)$ represent a stochastic diffusion process [1], and the motion of $z(t)$ is described by linear differential equations involving the control function u. The control function $u[t, y, z]$ is formed at each instant of time t on the basis of the realized values of $y(t)$ and $z(t)$. It is shown that the problem of bringing the point $z(t)$ into an ε-neighborhood of $y(T)$ $(T > 0)$ with a probability $p < 1$ has a finite solution $u[t, y, z]$ if the motion of $z(t)$ is completely controlled in a certain sense and the parameters of the process $y(t)$ are held within certain bounds. When the average value $M\{y(t)\}$ is described by linear equations, we obtain an explicit form of the control function u which is a linear function of y and z. Several optimal control problems are discussed incidentally. The problem is solved by the Liapunov function method [2,3] modernized for the present problem. This modernization makes use of concepts from the theory of dynamic programming [4].

1. Preliminary remarks. The present article considers two motions in n-dimensional space: a tracked random motion $y(t)$ and a tracking controlled motion $z(t)$. We shall describe these motions.

We shall assume that $y(t) = \{y_1(t), \ldots, y_n(t)\}$ describes an n-dimensional diffusion-type* stochastic process (see, for example, [1, pp. 247-262]). The $y(t)$ process can be described intuitively as follows.

* Reference [1] describes a scalar diffusion process ($n = 1$); nevertheless, here and in what follows we shall refer to [1] when we consider vector processes, since the transition from the case $n = 1$ to the case $n > 1$ requires no essential theoretical changes in the discussion.

The change $y(t_2) - y(t_1)$ is made up of small random increments $dy(t)$, each of which is a Gaussian random variable with a mean value and matrix of second moments of an order corresponding to the time interval dt.

In other words, during the time dt the point $y(t)$ is displaced by some amount mdt and is distributed according to the normal law, with the dispersion of the distribution proportional to dt.

For this reason we shall describe the process $y(t)$ by the "differential equation"

$$dy(t) = m(t,\ y(t))\,dt + R(t,\ y(t))\,dq(t) \tag{1.1}$$

Here m is a known n-dimensional vector function $\{m_i\}$, the matrix R is an n-by-n matrix $\{r_{ij}\}$, and $q(t)$ is an n-dimensional random vector whose components $q_i(t)$ are independent scalar processes of Brownian motion.

The processes $q_i(t)$ are consequently Gaussian processes with independent increments satisfying the conditions

$$M\{q_i(t_2) - q_i(t_1)\} = 0 \qquad (i = 1, \ldots, n) \tag{1.2}$$

$$M\{[q_i(t_2) - q_i(t_1)]\,[q_j(t_2) - q_j(t_1)]\} = |t_2 - t_1|\,\delta_{ij} \tag{1.3}$$
$$(\delta_{ij} = 0,\ i \neq j,\ \delta_{ii} = 1;\ i = 1, \ldots, n;\ j = 1, \ldots, n)$$

The symbol $M\{\alpha\}$ will be used to denote the mathematical expectation of the random variable α, and the symbol $M\{\alpha|\beta\}$ to denote the conditional mathematical expectation of α.

The elements r_{ij} of the matrix R define the matrix $\{\sigma_{ij}dt\}$ of second moments of the random distribution of the quantity $dy(t)$, namely

$$\sigma_{ij}(t, y) = \sum_{k=1}^{n} r_{ik}(t, y)\, r_{jk}(t, y) \tag{1.4}$$

The matrix $\{\sigma_{ij}\}$ is positive semi-definite.

Let us define more precisely the meaning of Equation (1.1), which is used for the intuitive description of the process $y(t)$. It is assumed throughout this article that the domain of variation of the argument t is the interval $[0,\ T]$ $(T > 0)$. Then Equation (1.1) can be interpreted strictly by means of a stochastic integral equation (see, for example, [1, pp. 250-258])

$$y(t) - y(0) = \int_0^t m(s, y(s))\,ds + \int_0^t R(s, y(s))\,dq(s) \tag{1.5}$$

Here the initial condition $y(0)$ is assumed to be fixed, and the integrals on the right have the meaning of stochastic integrals (see, for example, [1, pp. 392-404]).

We shall denote the norm of the vector g by the symbol $\|g\|$. Let the functions m and R be continuous and satisfy Lipschitz conditions in y:

$$|m_i(t, y^{(1)}) - m_i(t, y^{(2)})| \leqslant K\|y^{(1)} - y^{(2)}\| \qquad \begin{pmatrix} i=1, \ldots, n \\ j=1, \ldots, n \end{pmatrix}$$
$$|r_{ij}(t, y^{(1)}) - r_{ij}(t, y^{(2)})| \leqslant K\|y^{(1)} - y^{(2)}\|$$

and let them be bounded as $\|y\| \to \infty$.

$$\|m(t, y)\| \leqslant K(1 + \|y\|^2)^{1/2} \qquad \begin{pmatrix} i=1, \ldots, n \\ j=1, \ldots, n \end{pmatrix} \qquad (K = \mathrm{const})$$
$$|r_{ij}(t, y)| \leqslant K(1 + \|y\|^2)^{1/2}$$

Then there exists a random process $y(t)$ which satisfies Equation (1.5) for every $t \in [0, T]$ (with a probability of unity).

Almost all realizations $y(\omega, t)$ of this process $y(t)$ are continuous in the interval $[0, T]$ and can be continued over the entire interval in the sense that

$$M\{\max(\|y(\omega, t)\|^2,\ 0 \leqslant t \leqslant T)\} < \infty$$

Let the controlled motion of $z(t)$ be described by the linear vector differential equation

$$\frac{dz}{dt} = A(t)z + b(t)u \tag{1.6}$$

where $z(t) = z_1(t), \ldots, z_n(t)$ is an n-dimensional vector, $A(t)$ is an n-by-n matrix with continuous elements $a_{ij}(t)$, $b(t) = b_1(t), \ldots, b_n(t)$ is a continuous n-dimensional vector, and u is a scalar quantity which describes the controlling effect. If in Equation (1.6) u is a sufficiently smooth function, then Equations (1.1) and (1.6) give rise to a random process $\{y(t), z(t)\}$ for every initial condition $y(0)$, $z(0)$.

Let us clarify this assertion. If the function $u[t, y, z]$ is continuous, satisfies Lipschitz conditions in y and z

$$|u[t, y^{(1)}, z^{(1)}] - u[t, y^{(2)}, z^{(2)}]| \leqslant K\|\{y^{(1)}, z^{(1)}\} - \{y^{(2)}, z^{(2)}\}\| \tag{1.7}$$

and the boundedness condition

$$|u[t, y, z]| \leqslant K(1 + \|y\|^2 + \|z\|^2)^{1/2} \qquad (K = \mathrm{const}) \tag{1.8}$$

then there exists a $2n$-dimensional stochastic process $\{y(t), z(t)\}$ satisfying Equation (1.5) and the equation

$$z(t)-z(0)=\int^{t}[A(s)z(s)+b(s)u[s,y(s),z(s)]]\,ds \tag{1.9}$$

almost all realizations $\{y(t,\omega),\ z(t,\omega)\}$ of which are continuous for $t\in[0,\ T]$ and which is continuable over the whole interval $[0,\ T]$ in the sense that

$$M\{\max(\|y(\omega,t),\ z(\omega,t)\|^2),\quad 0\leqslant t\leqslant T\}<\infty \tag{1.10}$$

It is exactly this meaning that we will attach hereafter to the expression: "The process $\{y(t),\ z(t)\}$ is described by the equations (1.1) and (1.6)."

Control functions $u[t,\ y,\ z]$ satisfying the conditions (1.7) and (1.8) (for any K = const) will be called admissible. The set of admissible control functions u will be denoted by the symbol U.

The initial conditions $\{y(0),\ z(0)\}$ of the process $\{y(t),\ z(t)\}$ will be assumed to be fixed (random values identically equal to $y(0),\ z(0)$).

We shall use the symbol $P[\alpha;\ y(0),\ z(0)]$ to denote the probability of events α related to the random variables $y(t),\ z(t)$ with initial conditions $y(0),\ z(0)$, and we shall use the symbol $P[\alpha|\beta;\ y(0),\ z(0)]$ to denote the conditional probabilities of these events. Also, if it should become necessary to emphasize the role played by the initial conditions, we shall write the symbol for mathematical expectation as $M\{\alpha;\ y(0),\ z(0)\}$ or $M\{\alpha|\beta,\ y(0),\ z(0)\}$. Where the meaning is not in doubt, the symbols $y(0)$ and $z(0)$ will be omitted in the notation for P and M.

The fundamental result of the article will be formulated for the case when the quantity $m(t,\ y)$ is a linear function of y, that is,

$$m(t,y)=M(t)y \tag{1.11}$$

Here $M(t)$ is a continuous n-by-n matrix function $\{m_{ij}(t)\}$. In this case we may suppose that $M\ y(t)\ |\ y(\tau)=\eta\}$, the average of $y(t)$ for $t\geqslant\tau$, satisfies the equation

$$\frac{dM\{y(t)\,|\,y(\tau)=\eta\}}{dt}=M(t)M\{y(t)\,|\,y(\tau)=\eta\} \tag{1.12}$$

obtained from (1.1) by averaging. We shall denote by the symbol $M\{t,\ \tau\}$ the fundamental matrix of solutions of the system (1.12) ($M\{\tau,\ \tau\}=E$). Then

$$M\{y(t)\,|\,y(\tau)=\eta\}=M(t,\tau)\eta \tag{1.13}$$

In particular, if $M(t) = M = \text{const}$ for $t \in [0, T]$, then

$$M\{y(t) \mid y(\tau) = \eta\} = e^{M(t-\tau)}\eta \tag{1.14}$$

2. Statement of the problem. Let us consider the process $\{y(t), z(t)\}$, described by Equations (1.1) and (1.6) for $u \in U$.

The problem consists in the choice of a control function $u^\circ \in U$ for which the motion of $z(t)$ at a given moment $t = T > 0$ is brought into a neighborhood of the point $y(T)$. Stated more exactly, for given values $T > 0$, $\varepsilon > 0$, $N > 0$ and $p < 1$ it is required to find a function $u = u^\circ[t, y, z] \in U$ such that when this function is substituted into Equation (1.9), the stochastic process $\{y(t), z(t)\}$ described by Equations (1.5) and (1.9) will satisfy the condition

$$P[\|y(T) - z(T)\| < \varepsilon;\ y(0),\ z(0)] > p \tag{2.1}$$

for all initial conditions

$$\|y(0)\| \leqslant N, \qquad \|y(0) - z(0)\| \leqslant N \tag{2.2}$$

One more restriction should be added to the problem conditions. Since the control resources must be assumed to be limited, we shall require the desired control function u° to satisfy the restriction

$$\int_0^T M\{(u^\circ[t, y(t),\ z(t)])^2;\ y(0),\ z(0)\}\, dt < \infty \tag{2.3}$$

if $\|y(0)\| \leqslant N$, $\|y(0) - z(0)\| \leqslant N$.

Let us clarify the meaning of the problem. The control function $u = u^\circ \in U$ is chosen as a function of the variables t, y and z. This means that the value of the control function is defined at each instant of the process $\tau \in [0, T]$ on the basis of measurement of the realized values $y(\tau)$, $z(\tau)$. It is assumed, consequently, that in the control process it is possible to measure the values of $y(\tau)$ and $z(\tau)$ and instantaneously transmit a signal giving the results of the measurement to a control apparatus which will generate an output $u = u^\circ[\tau, y(\tau), z(\tau)]$. The future values of $y(t)$ and $z(t)$ for $t > \tau$ are unknown, but the stochastic prognosis of $y(t)$ and $z(t)$ is taken into account in defining the function $u^\circ[\tau, y(\tau), z(\tau)]$. Such a statement of the problem is in agreement with optimal control problems involving measurable coordinates and random disturbances. Such problems have been considered, for example, in [5-10].

It should be noted that the control function u° which brings $z(t)$

into a neighborhood of $y(t)$ cannot be chosen in our case by a simple pursuit rule which at each instant of time t points the velocity vector $dz(t)/dt$ in the direction of $y(t)$. If such a pursuit were realizable, the solution of the problem would be considerably simplified. In the present case, however, u is a scalar quantity, so that at time t, by varying $u(-\infty < u < \infty)$, we can rotate the vector $dz(t)/dt$ only within the limits of a half-plane formed by the vectors $A(t)z(t)$ and $b(t)u$, which does not, in general, contain the point $y(t)$. A simple pursuit ($dz(t)$ aimed at $y(t)$) would be possible if Equation (1.6), instead of a term $b(t)u$, contained a term $B(t)u$, where $B(t)$ is a nonsingular n-by-n matrix and u is an n-dimensional vector.

We shall transform the problem formulated above into a form which is suitable for further consideration. By the change of variables $x = z - y$, $y = y$, Equation (1.6) is transformed (taking Equation (1.1) into account) into the equation

$$dx(t) = [A(t)(x(t) + y(t)) - m(t, y(t)) + b(t)u]dt - R(t, y(t))dq(t) \quad (2.4)$$

Now the problem is formulated in the following manner:

Problem A. Given the values $T > 0$, $\varepsilon > 0$, $N > 0$ and $p < 1$, it is required to find an admissible control function $u^{\circ}[t, x, y]$ satisfying the condition (2.3) and such that if $u = u^{\circ}$ in Equation (2.4), the condition

$$P[\|x(T)\| < \varepsilon;\ x(0),\ y(0)] > p \quad (2.5)$$

is satisfied if

$$\|x(0)\| \leqslant N, \qquad \|y(0)\| \leqslant N \quad (2.6)$$

If a rigorous description is used, Equation (2.4) may be interpreted as the stochastic integral equation

$$x(t) - x(0) = \int_0^t [A(s)(x(s) + y(s)) - m(s, y(s)) + b(s)u[s, x(s), y(s)]]\, ds - \\ - \int_0^t R[s, y(s)]\, dq(s) \quad (2.7)$$

We should note, in conclusion, that we are considering the problem of efficiently defining a control function u°. Therefore, in what follows we shall not try to calculate a solution which is (in some sense) the best solution. For this same reason, we shall restrict ourselves to the linear case. All of the reasoning may be applied to more general nonlinear cases, but an efficient calculation of $u^{\circ}[t, x, y]$ is difficult in nonlinear cases.

Finally, we must emphasize that the statement of the problem assumes that the functions $m(t, y)$, $R(t, y)$, $A(t)$ and $b(t)$ are known and satisfy the conditions stated in Section 1.

3. Fundamental result. Let us formulate a theorem on the control function u° which provides a solution of Problem A.

Let us suppose that the system prescribed by the equation

$$dx/dt = A(t)x + b(t)u \tag{3.1}$$

will be completely controllable in the interval $[0, T]$. This means that for any point $x^{(1)}$, $x^{(2)}$ and for the interval $[t_1, t_2] \in [0, T]$, $t_2 > t_1$, there exists a control function u bringing the point $x^{(1)}$ to the point $x^{(2)}$ during the time $t_1 \leqslant t \leqslant t_2$. In other words, there exists a continuous (or sectionally continuous) function $u(t)$ such that if it is substituted into Equation (3.1), one of its solutions $x(t)$ will satisfy the conditions $x(t_1) = x^{(1)}$, $x(t_2) = x^{(2)}$.

Let us derive a sufficient condition for controllability. We shall use the sybol $F(t, \tau) = \{f_{ij}(t, \tau)\}$ to denote the fundamental matrix of solutions of the system (3.1) ($f_{ij}(\tau, \tau) = \delta_{ij}$), the symbol $F^{-1}(t, \tau) = G(t, \tau) = \{g_{ij}(t, \tau)\}$ to denote the inverse of the matrix F, and the symbol $h(t, \tau) = \{h_i(t, \tau)\}$ to denote the vector $G(t, \tau)b(t)$.

In accordance with the results of an earlier article [11, pp.964-966], a sufficient condition for complete controllability of the system (3.1) is that the functions $h_i(t, \tau)$ be completely linearly independent (with respect to t for fixed τ, that is, that the linear combination $\lambda_1 h_1(t, \tau) + \ldots + \lambda_n h_n(t, \tau)$ should not vanish identically anywhere in the interval $\tau < t < T$ if $\lambda_1{}^2 + \ldots + \lambda_n{}^2 \neq 0$.

For complete linear independence of the functions $\{h_i\}$ it is sufficient [12] that the vectors $L_1(t), \ldots, L_n(t)$ be linearly independent for all $t \in [0, T]$, (almost all $t \in [0, T]$). Here

$$L_1(t) = b(t), \quad L_{k+1}(t) = \frac{dL_k(t)}{dt} - A(t)L_k(t) \qquad (k = 1, \ldots, n-1)$$

We shall denote the condition of independence of the vectors $L_1(t), \ldots, L_n(t)$ by condition L.

It should be noted that the matrix $G^*(t, \tau) = S(t)$ satisfies the equation

$$dS/dt = -A^*(t)S \tag{3.2}$$

where the symbol * denotes the transpose; therefore, the expression $\lambda_1 h_1(t, \tau) + \ldots + \lambda_n h_n(t, \tau)$ may be considered as the scalar product $(\psi(t) \times b(t))$, which is of fundamental importance in Pontriagin's maximum principle [10]. Here the vector

$$\psi(t) = \{\psi_j(t)\} = \left\{\sum_{i=1}^{n} \lambda_i g_{ij}\right\}$$

is a solution of the equation

$$d\psi / dt = -A^*(t)\,\psi \tag{3.3}$$

Consequently, it is sufficient for complete controllability of the system (3.1) that the scalar product $(\psi(t) \times b(t))$ should not vanish identically for any nontrivial solution $\psi(t)$ of Equation (3.3).

It should be noted that the concept of controllability of a linear system has been considered by a number of authors from various viewpoints [10-14].

The following statement is true.

Theorem 3.1. In the interval $[0, T]$ let the controllability condition L be satisfied, let the functions $A(t)$, $b(t)$, $M(t)$, and $R(t, y)$ be continuous; let the function $R(t, y) = \{r_{ij}(t, y)\}$ satisfy the Lipschitz conditions in y

$$|r_{ij}(t, y^{(1)}) - r_{ij}(t, y^{(2)})| \leqslant K \| y^{(1)} - y^{(2)} \| \tag{3.4}$$

and be bounded so that

$$|\sigma_{ij}(t, y)| \leqslant \sigma^2(t) \tag{3.5}$$

(the function $\sigma^2(t)$ is continuous if $t \in [0, T]$).

Then for any numbers $\varepsilon > 0$, $N > 0$ and $p < 1$ it is possible to construct an admissible control function $u = u^\circ[t, x, y]$ which satisfies the conditions of Problem A. This control function may be chosen as a linear function

$$u^\circ[t, x, y] = \sum_{i=1}^{n} [\mu_i(t)\, x_i + \nu_i(t)\, y_i] \tag{3.6}$$

where $\mu_i(t)$, $\nu_i(t)$ are continuous functions.

Determining the functions $\mu_i(t)$ and $\nu_i(t)$ reduces to solving the Cauchy problem with known initial conditions for a system of ordinary differential equations with the argument $t \in [0, T]$. The coefficients of these equations are determined by the functions $A(t)$, $b(t)$, $M(t)$ and therefore Problem A is solved as efficiently as the Cauchy problem is solved.

4. Method of solution. To solve Problem A we shall use a method similar to the Liapunov function method in the theory of stability of motion [2,3].

Let us transform Problem A somewhat. From the Chebyshev inequality [1]

$$p[\|x\| \geqslant \varepsilon] \leqslant \frac{M\{\|x\|^2\}}{\varepsilon^2} \tag{4.1}$$

to satisfy condition (2.5) it is sufficient that the inequality

$$M\{\|x(T)\|^2;\ x(0), y(0)\} < (1-p)\varepsilon^2 \qquad (\|x(0)\| \leqslant N,\ \|y(0)\| \leqslant N) \tag{4.2}$$

be satisfied.

In solving Problem A, therefore, we shall hereafter look for a control function u° satisfying not condition (2.5) but the somewhat stronger requirement (4.2). The Chebyshev inequality (4.1) gives a crude estimate of the value of $p[\|x\| \geqslant \varepsilon]$, but the replacement of condition (2.5) by condition (4.2) is justified by the fact that we are considering not the best control function but an efficiently calculable control function u° satisfying the conditions of Problem A.

Let us formulate a sufficient condition for u°.

Theorem 4.1. If it is possible to demonstrate a continuously differentiable function $v[t, x, y]$ which has the form

$$v[t,x,y] = \sum_{i,j=1}^{n} [\alpha_{ij}(t) x_i x_j + 2\beta_{ij}(t) x_i y_j + \gamma_{ij}(t) y_i y_j] + \lambda(t) \qquad (t \in [0,T]) \tag{4.3}$$

satisfies the conditions

$$v[0,x,y] < (1-p)\varepsilon^2 \qquad (\|x\| \leqslant N,\ \|y\| \leqslant N) \tag{4.4}$$

$$v[T,x,y] = \|x\|^2 \tag{4.5}$$

and has a derivative $(dM\{v\}/dt;\ u^{\circ})$ [7,9] which for $u = u^{\circ}[t, x, y]$ is non-positive by virtue of Equations (1.1) and (2.4), with

$$\left(\frac{dM\{v\}}{dt};\ u^{\circ}\right) \leqslant -c(u^{\circ})^2 \qquad (c > 0 - \text{const}) \tag{4.6}$$

then the control function u° satisfies the conditions of Problem A.

Let us recall that $(dM\{v\}/dt;\ u^{\circ})$ means $(t = \tau,\ x = \xi,\ y = \eta)$:

$$\left(\frac{dM\{v\}}{dt}\right) = \lim \left\{\frac{1}{t-\tau}[M\{v[t, x(t), y(t)] - v[\tau, x(\tau), y(\tau)] \mid x(\tau)=\xi, y(\tau)=\eta\right\}$$

$$\text{as } t \to \tau + 0 \tag{4.7}$$

and is defined by the infinitesimal generating operator [15] of the process $\{x(t),\ y(t)\}$.

Proof of Theorem 4.1. The stochastic process $\{x(t),\ y(t)\}$, described by Equations (1.5) and (2.7), is continuable over the interval $[0,\ T]$ and satisfies the conditions (see, for example, [1, pp.251,257])

$$M\{\|x(t)\|^2 + \|y(t)\|^2\} < \infty \tag{4.8}$$

$$M\{x(t) - x(\tau) \mid x(\tau) = \xi,\ y(\tau) = \eta\} = (t-\tau)[A(\tau)(\xi + \eta) - \\ - m(\tau, \eta) + b(\tau)u[\tau, \xi, \eta]] + o(t-\tau)(1 + \|\xi\|^2 + \|\eta\|^2)^{1/2} \tag{4.9}$$

$$M\{y(t) - y(\tau) \mid y(\tau) = \eta\} = m(\tau, \eta)(t-\tau) + o(t-\tau)(1 + \|\eta\|^2)^{1/2} \tag{4.10}$$

$$M\{[x_i(t) - x_i(\tau)][x_j(t) - x_j(\tau)] \mid x(\tau) = \xi,\ y(\tau) = \eta\} = \\ = (\tau - t)\sigma_{ij}(\tau, \eta) + (1 + \|\xi\|^2 + \|\eta\|^2)\, o(t-\tau) \tag{4.11}$$

$$M\{[x_i(t) - x_i(\tau)][y_j(t) - y_j(\tau)] \mid x(\tau) = \xi,\ y(\tau) = \eta\} = \\ = (\tau - t)\sigma_{ij}(\tau, \eta) + (1 + \|\xi\|^2 + \|\eta\|^2)\, o(t-\tau) \tag{4.12}$$

$$M\{[y_i(t) - y_i(\tau)][y_j(t) - y_j(\tau)] \mid x(\tau) = \xi,\ y(\tau) = \eta\} = \\ = (t-\tau)\sigma_{ij}(\tau, \eta) + (1 + \|\eta\|^2)\, o(t-\tau) \tag{4.13}$$

Here the symbol $o(t - \tau)$ denotes a quantity which tends to zero faster than $t - \tau$, and the estimate $o(t - \tau)$ is uniform with respect to ξ, η, $\tau \in [0,\ T]$.

Calculating the derivative $(dM\{v\}/dt;\ u)$ by virtue of Equations (1.1) and (2.4) (or, equivalently, by virtue of Equations (1.5) and (2.7)) and taking Equations (4.9) to (4.13) into account, we obtain* (at the point $(t = \tau,\ x(\tau) = \xi,\ y(\tau) = \eta)$

* The calculations are omitted here. The order of calculating $dM\{v\}/dt$, starting from the intuitive description of the stochastic process, is discussed, for example, in [9].

$$\left(\frac{dM\{v\}}{dt};\ u\right) = \sum_{i=1}^{n} \left\{\frac{\partial v}{\partial x_j}[a_{ij}(\tau)(\xi_j + \eta_j) - m_i(\tau, \eta) + b_i(\tau)u] + \frac{\partial v}{\partial y_i} m_i(\tau, \eta)\right\} +$$
$$+ \frac{\partial v}{\partial t} + \frac{1}{2}\sum_{i,j=1}^{n}\left\{\frac{\partial^2 v}{\partial x_i \partial x_j} - 2\frac{\partial^2 v}{\partial x_i \partial y_j} + \frac{\partial^2 v}{\partial y_i \partial y_j}\right\}\sigma_{ij}(\tau, \eta) \tag{4.14}$$

It should be noted that as a result of Equations (4.9) to (4.13), for each admissible control function u the following estimate is also valid

$$M\{v[t, x(t), y(t)] - v[\tau, x(\tau), y(\tau)] \mid x(\tau) = \xi, y(\tau) = \eta\} =$$
$$= (t-\tau)\left(\frac{dM\{v\}}{dt};\ u\right) + o(t-\tau)(1 + \|\xi\|^2 + \|\eta\|^2) \tag{4.15}$$

We set up the quantity

$$V[t; x(0), y(0)] = M\{v[t, x(t), y(t)];\ x(0), y(0)\} \tag{4.16}$$

Since the process $\{x(t, y(t)\}$ is continuable over $[0, T]$ and satisfies condition (4.8), it follows that the quantity (4.16) is defined and finite for all values $t \in [0, T]$.

Let us calculate the change $V(t) - V(\tau)$. From the formula for repeated conditional mathematical expectations (see, for example, [1, pp.38-40]), we have

$$V[t; x(0), y(0)] - V[\tau; x(0), y(0)] =$$
$$= M\{M\{v[t, x(t), y(t)] - v[\tau, x(\tau), y(\tau)] \mid x(\tau), y(\tau)\};\ x(0), y(0)\} \tag{4.17}$$

Taking the estimates (4.8) and (4.15) into consideration, we can write the equality

$$V[t; x(0), y(0)] - V[\tau; x(0), y(0)] = \tag{4.18}$$
$$= (t-\tau) M\left\{\left(\frac{dM\{v\}}{dt};\ u\right);\ x(0), y(0)\right\} + o(t-\tau)D \qquad (D = \text{const})$$

From (4.18) we conclude that the quantity $V[t;\ x(0),\ y(0)]$ is continuous with respect to time t and that (for $u = u^\circ$) the relations

$$\frac{dV[t; x(0), y(0)]}{dt} = M\left\{\left(\frac{dM\{v\}}{dt};\ u^\circ\right)\right\} \leqslant -cM\{(u^\circ)^2\} \leqslant 0 \tag{4.19}$$

are valid.

Integrating (4.19) with respect to t between the limits $t = 0$ and $t = T$, and remembering that

$$V[0;\ x(0),\ y(0)] = v[0,\ x(0),\ y(0)]$$

$$V[T;\ x(0),\ y(0)] = M\{v[T,\ x(T),\ y(T)];\ x(0),\ y(0)\} = M\{\|x(T)\|^2\}$$

by virtue of (4.5), we obtain the inequality

$$M\{\|x(T)\|^2\} \leqslant v[0,\ x(0),\ y(0)] \tag{4.20}$$

If $\|x(0)\| \leqslant N$, $\|y(0)\| \leqslant N$, then by virtue of (4.4) we obtain the inequality

$$M\{\|x(T)\|^2;\ x(0),\ y(0)\} < (1-p)\,\varepsilon^2$$

which proves the validity of the theorem.

Thus, in order to solve the problem, we must find functions v and u^0 which satisfy the conditions of Theorem 4.1. Like the problem of finding the Liapunov function in the theory of stability of motion, this problem is highly indefinite. However, in the present case we shall compare Problem A with a problem in optimal control, in the solution of which the Liapunov function v and the control function $u = u^0$ are uniquely defined. It is worth noting that such a method may also be useful sometimes in finding the Liapunov function for problems in the theory of stability of motion.

5. Supplementary material from the theory of optimal control. Let us mention some supplementary facts which will be used in the following section for the proof of Theorem 3.1. We consider the system of equations

$$\frac{dy}{dt} = M(t)\,y, \qquad \frac{dx}{dt} = A(t)\,x + [A(t) - M(t)]\,y + b(t)\,u \tag{5.1}$$

which coincides with Equations (1.1) and (2.4) if we eliminate from the latter the random terms $R(t,\ y)\ dq(t)$.

We formulate an optimal control problem for the system (5.1).

Problem B(c). It is required to determine a control function $u = u^*[c;\ t,\ x,\ y]$ such that for each initial condition $x(\tau)$, $y(\tau)$ ($\tau \in [0,T]$) the functional

$$J[c;\ x(\tau),\ y(\tau);\ u] = \int_\tau^T cu^2[c;\ t,\ x(t),\ y(t)]\,dt + \|x(T)\|^2 \tag{5.2}$$

takes on the minimum possible value.

Problem B(c) is the problem of analytically designing [16] an optimal control function u, but here we are considering the problem, unlike [16], for a finite interval of time $[0, T]$. Similar problems have been considered, for example, in [17] and [18].

Let us describe a solution of Problem B(c) based on the method of dynamic programming.

Let the functions $w[c;\ t,\ x,\ y]$ and $u^*[c;\ t,\ x,\ y]$ satisfy the conditions

$$\left(\frac{dw}{dt};\ (5.1),\ u^*\right) + c\,[u^*[c;\ t,\ x,\ y]]^2 = \min\left[\left(\frac{dw}{dt};\ (5.1),\ u\right) + cu^2\right] = 0 \qquad (5.3)$$

$$w\,[c,\ T,\ x,\ y] = \|x\|^2 \qquad (5.4)$$

Here the symbol $(dw/dt;\ (5.1),\ u)$ stands for the derivative of the function w by virtue of the system (5.1) for a chosen u.

Then $u^*[c;\ t,\ x,\ y]$ is an optimal control function satisfying the conditions of Problem B(c), and we have the equality

$$w\,[c;\ \tau,\ x(\tau),\ y(\tau)] = \min J\,[c;\ x(\tau),\ y(\tau);\ u] = J\,[c;\ x(\tau),\ y(\tau);\ u^*] \qquad (5.5)$$

For completeness of presentation, we shall give a proof of this statement. For $u = u^*$, from (5.3), integrating with respect to t from $t = \tau$ to $t = T$ and taking (5.4) into account, we obtain the equality

$$w\,[c;\ \tau,\ x(\tau),\ y(\tau)] = \int_\tau^T c\,[u^*[c;\ t,\ x(t),\ y(t)]]^2\,dt + \|x(T)\|^2 \qquad (5.6)$$

For $u \neq u^*$, from (5.3) we obtain the inequality

$$\left(\frac{dw}{dt};\ (5.1),\ u\right) \geqslant -cu^2 \qquad (5.7)$$

Integrating this inequality, we obtain the inequality

$$w\,[c;\ \tau,\ x(\tau),\ y(\tau)] \leqslant \int_\tau^T cu^2(t)\,dt + \|x(T)\|^2 \qquad (5.8)$$

The inequality (5.7) and the inequality (5.8) together prove the validity of the statement that u^* is an optimal function.

The functions w and u^* satisfying conditions (5.3) and (5.4) exist and are of the form

$$w[c;\, t,\, x,\, y] = \sum_{i,\, j=1}^{n} [\alpha_{ij}(c,\, t)\, x_i x_j + 2\beta_{ij}(c,\, t)\, x_i y_j + \gamma_{ij}(c,\, t)\, y_i y_j] \tag{5.9}$$

$$u^*[c;\, t,\, x,\, y] = \sum_{i=1}^{n} [\mu_i(c,\, t)\, x_i + \nu_i(c,\, t)\, y_i] \tag{5.10}$$

Let us prove this. We set up equations for the coefficients α_{ij}, β_{ij} γ_{ij}, μ_i, ν_i, obtained from the conditions of (5.3). For this purpose we must eliminate u^* from (5.3), making use of the fact that for $u = u^*$ there is a minimum, that is, the equality

$$\left(\frac{\partial}{\partial u}\left[\left(\frac{dw}{dt};\ (5.1),\ u\right) + cu^2\right]\right)_{u^*} = 0 \tag{5.11}$$

must be satisfied.

After carrying out the indicated operations we obtain the following equations for w and u^*:

$$\frac{\partial w}{\partial t} + \sum_{i=1}^{n}\left[\frac{\partial w}{\partial x_i}(a_{ij}(t)\, x_j + (a_{ij}(t) - m_{ij}(t))\, y_j) + \frac{\partial w}{\partial y_i} m_{ij}(t)\, y_j\right] - \\ - \frac{1}{4c}\left[\sum_{i=1}^{n} b_i(t)\frac{\partial w}{\partial x_i}\right]^2 = 0 \tag{5.12}$$

$$2cu^* + \sum_{i=1}^{n} b_i(t)\frac{\partial w}{\partial x_i} = 0 \tag{5.13}$$

On the left side of (5.12) we equate the coefficients of identical expressions $x_i x_j$, $x_i y_j$, $y_i y_j$ to zero, thus obtaining a system of equations from which we can determine the coefficients $\alpha_{ij}(c,\, t)$, $\beta_{ij}(c,\, t)$, $\gamma_{ij}(c,\, t)$. Let us write this system in normal form

$$\begin{aligned}
\frac{d\alpha_{ij}}{dt} &= \frac{1}{c}\sum_{k,\, l=1}^{n} \alpha_{ki}\alpha_{lj} b_k(t)\, b_l(t) - \sum_{k=1}^{n} (\alpha_{ik} a_{kj}(t) + \alpha_{jk} a_{ki}(t)) \\
\frac{d\beta_{ij}}{dt} &= \frac{1}{c}\sum_{k,\, l=1}^{n} \alpha_{ki}\beta_{lj} b_k(t)\, b_l(t) - 2\sum_{k=1}^{n} [\alpha_{ik}(a_{kj} - m_{kj}) + \beta_{ik} m_{kj}] \\
\frac{d\gamma_{ij}}{dt} &= \frac{1}{c}\sum_{k,\, l=1}^{n} \beta_{ki}\beta_{lj} b_k(t)\, b_l(t) - \\
&\quad - \sum_{k=1}^{n} [2\beta_{ki}(a_{kj}(t) - m_{kj}(t)) + \gamma_{ik} m_{kj}(t) + \gamma_{jk} m_{ki}(t)]
\end{aligned} \tag{5.14}$$

The system of Equations (5.14) is to be solved in the internal $[0,\ T]$

for the following initial conditions:

$$\alpha_{ij}(c, T) = \delta_{ij}, \quad \beta_{ij}(c, T) = 0, \quad \nu_{ij}(c, T) = 0 \qquad (\delta_{ii} = 1, \ \delta_{ij} = 0, \ i \neq j) \quad (5.15)$$

as found from the boundary condition (5.4) for $w[c;\ t,\ x,\ y]$.

It can be shown that the system of Equations (5.14) has a unique solution satisfying the conditions (5.15) and continuable over the entire interval $[0,\ T]$, for any number $c > 0$. Indeed, the right-hand parts of the Equations (5.14) satisfy the conditions for the local existence and uniqueness of solutions. It is therefore sufficient to show that the solutions $\alpha_{ij}(c,\ t)$, $\beta_{ij}(c,\ t)$, $\gamma_{ij}(c,\ t)$ which exist at the point (5.15) can be continued in t over the entire $[0,\ T]$ in the direction of decreasing t. Since the right-hand parts of (5.14) contain second powers of the unknown functions α_{ij}, β_{ij}, γ_{ij}, it is necessary to show this.

To prove the continuability of the solutions α_{ij}, β_{ij}, γ_{ij}, we note that the quadratic form $w[c;\ t,\ x,\ y]$ is uniformly bounded (for $\|x\|^2 + \|y\|^2 = 1$) for all those values $t \in (\vartheta,\ T)$ for which a solution exists for the system (5.14) with the initial condition (5.15). Indeed, from the definition of the quantity w, the inequalities

$$\begin{gathered} w[c;\, t, x, y] > 0 \\ w[c;\, \tau, x(\tau), y(\tau)] = \min_u J[c;\, x(\tau), y(\tau);\, u] \leqslant \\ \leqslant J[c;\, x(\tau), y(\tau);\, u = 0] \leqslant Q = \text{const} \\ \|x(\tau)\|^2 + \|y(\tau)\|^2 = 1 \qquad \text{if} \quad \tau \in (\vartheta, T] \end{gathered} \tag{5.16}$$

are satisfied.

From the inequalities (5.16) it follows that the solutions of (5.14), (5.16) are uniformly bounded and therefore continuable over the entire interval $[0,\ T]$.

Our assertion is proved.

Thus, there exists a function $w[c;\ t,\ x,\ y]$ which satisfies the conditions (5.3) and is expressed by the quadratic form (5.9). From (5.9) and the equality (5.11) it follows that the control function u^* satisfies the equation

$$2cu^* + 2\sum_{i=1}^{n} [\alpha_{ij}(c,\, t)\, x_j + \beta_{ij}(c,\, t)\, y_j]\, b_i(t) = 0$$

that is,

$$u^*[c;\, t,\, x,\, y] = -\sum_{i=1}^{n} b_i(t)\,[\alpha_{ij}(c,\, t)\, x_j + \beta_{ij}(c,\, t)\, y_j] \tag{5.17}$$

and consequently, by virtue of (5.10),

$$\mu_j(c,\, t) = -\sum_{i=1}^{n} b_i(t)\,\alpha_{ij}(c,\, t), \qquad \nu_j(c,\, t) = -\sum_{i=1}^{n} b_i(t)\,\beta_{ij}(c,\, t) \tag{5.18}$$

For the subsequent discussion it is important that as the quantity $c > 0$ decreases to zero, the function $w[c;\ 0,\ x,\ y]$ also decreases to zero (if $||x||^2 + ||y||^2 = \text{const}$).

In fact, the following statement is true:

Lemma 5.1. In the interval $[0,\ T]$ let the system

$$dx\,/\,dt = A(t)\, x + b(t)\, u \tag{5.19}$$

satisfy the condition L (p. 321) for complete controllability. Then for any number $\Delta > 0$ it is possible to find a number $\delta > 0$ such that

$$w[c;\ 0,\ x,\ y] < \Delta\,(\|x\|^2 + \|y\|^2) \tag{5.20}$$

$$\int_0^T \{\max\,(w[c;\ t,\ x,\ y] \quad \text{for } \|x\|^2 + \|y\|^2 = 1)\}\, dt < \Delta \tag{5.21}$$

provided $o < c < \delta$.

Proof. Let a number $\Delta > 0$ be given. We select a number ϑ such that the inequalities

$$w[c;\, t,\, x,\, y] < (\|x\|^2 + \|y\|^2)\,\zeta \qquad (0 < c \leqslant 1,\ t \in [T - \vartheta,\ T]), \qquad \vartheta\zeta < \frac{\Delta}{4} \tag{5.22}$$

are satisfied.

Here ζ is some fixed number greater than unity. It is possible to select a number $\vartheta > 0$. Indeed,

$$w[c;\ t,\ x,\ y] \leqslant J[c;\ t,\ x,\ y;\ u = 0] = \|x(x,\ y,\ t,\ T)\|^2 \tag{5.23}$$

where $x(x,\ y,\ t,\ \tau)$ is the solution of the system (5.1) resulting from the initial conditions $x(t) = x,\ y(t) = y$ (for $t \leqslant \tau \leqslant T,\ u = 0$).

The solutions of the system (5.1) for $u = 0$ satisfy the inequality [19, p.23]

$$\|x(x,\,y,\,t,\,\tau)\|^2 \leqslant (\|x\|^2 + \|y\|^2)\, e^{4nK(\tau - t)} \tag{5.24}$$

where $K = \max\,\{|a_{ij}(t)|,\ |m_{ij}(t)|\}$; the inequality

$$w\,[c;\,t,\,x,\,y] \leqslant (\|x\|^2 + \|y\|^2)\, e^{4nK(\tau - t)} \tag{5.25}$$

therefore follows from (5.23).

Equation (5.25) proves that a number ϑ may be chosen and gives an estimate for it. We shall therefore assume that such a number $\vartheta > 0$ has been chosen.

Let us consider an auxiliary problem.

Problem C(τ). Given an instant of time $\tau \in [0,\ T]$ and a point $x_0 y_0 (\|x\|^2 + \|y\|^2 = 1)$. It is required to find the control function $u\tau(t)$ such that the solution $x(t)$, $y(t)$ of the system (5.1) satisfies the conditions

$$x(\tau) = x_0, \qquad y(\tau) = y_0, \qquad x(T) = 0 \tag{5.26}$$

$$\int_\tau^T u_\tau^2(t)\,dt = \min \tag{5.27}$$

Let us briefly outline the solution of Problem C(τ) in accordance with the method indicated for the solution of such problems in [20]. The solution $x(t)$ of the second equation of (5.1) can be written by Cauchy's formula (for $t = T$) as

$$x(T) = F(T,\tau)\,x_0 + \int_\tau^T F(T,\,\tau)\,F^{-1}(t,\,\tau)\,\{[A(t) - M(t)]\,y(t) + b(t)\,u(t)\}\,dt \tag{5.28}$$

where $F(t,\ \tau)$ is the fundamental matrix ($F(\tau,\ \tau) = E$) of the system (5.19) (for $u = 0$). Since $y(t) = M(t,\ \tau)y_0$, where $M(t,\ \tau)$ is the fundamental matrix ($M(\tau,\ \tau) = E$) of the first equation of (5.1) and since we must have $x = 0$ at time $t = T$, the equation for $u_\tau(t)$ follows from (5.28) (after multiplying (5.28) by $F^{-1}(T,\ \tau)$)

$$x_0 + \int_\tau^T F^{-1}(t,\,\tau)\,\{[A(t) - M(t)]\,M(t,\,\tau)\,y_0 + b(t)\,u_\tau(t)\}\,dt = 0 \tag{5.29}$$

where the condition (5.27) must be satisfied. The solution of the problem (5.27), (5.29) is described in [20] For complete controllability of the system (5.19), a solution $u_\tau(t)$ of the problem exists and is defined from the conditions

$$u_\tau(t) = \lambda^2 [x_0, y_0, \tau] \sum_{i=1}^{n} \lambda_i^\circ h_i(t, \tau) \tag{5.30}$$

Here

$$\lambda^{-2} [x_0, y_0, \tau] = \min \left\{ \int_\tau^T \left(\sum_{i=1}^{n} \lambda_i h_i(t, \tau) \right)^2 dt \right\} \quad \text{if} \quad \sum_{i=1}^{n} f_i \lambda_i = 1 \tag{5.31}$$

$$f = x_0 + \left[\int_\tau^T F^{-1}(t, \tau) [A(t) - M(t)] M(t, \tau)\, dt \right] y_0 \tag{5.32}$$

$$h(t, \tau) = F^{-1}(t, \tau)\, b(t)$$

and the λ_i° are the solutions of the problem (5.31), (5.32). The problem (5.31), (5.32) is the problem of finding the minimum of a quadratic form (5.31) in the variables λ_i under the linear condition (5.32). The solution of such a problem is well known.

For the rest of the discussion it is important that the quantity (5.37) is uniformly bounded for $\tau \in [0, T - \vartheta]$, $\|x_0\|^2 + \|y_0\|^2 = 1$ for every $\vartheta > 0$. Indeed, from (5.30) and (5.31) it follows that

$$\min \left[\int_\tau^T u^2(t)\, dt \right] = \int_\tau^T u_\tau^2(t)\, dt = \lambda^2 [x_0, y_0, \tau] \tag{5.33}$$

But the quantity $\lambda^2 [x_0, y_0, \tau]$ is uniformly bounded above in the interval $\tau \in [0, T - \vartheta]$, since the quantity (5.31) is bounded below by a positive number for $\tau \in [0, T - \vartheta]$. Indeed, this quantity is continuous with respect to τ, and by virtue of the complete controllability it is positive for all $\tau \in (0, T)$.

We now turn directly to the proof of Lemma 5.1.

We select a number $\delta > 0$ small enough to satisfy the inequality

$$\delta \max [T, 1] < \min \left[\frac{\Delta}{4\lambda^2} \text{ for } \tau \in [0, T - \vartheta] \right] \tag{5.34}$$

Since the inequality

$$w[c; \tau, x_0, y_0] = \min J[c; x(\tau)]$$

$$[y(\tau); u] \leqslant J[c; x(\tau), y(\tau), u_\tau] = c\lambda^2 [x(\tau), y(\tau), \tau] \qquad (x(\tau) = x_0, y(\tau) = y_0)$$

is valid, it follows that for $\|x\|^2 + \|y\|^2 = 1$, $0 \leqslant t \leqslant T - \vartheta$, and for

$c < \delta$ the inequalities

$$w[c; t, x, y] < \frac{\Delta}{4T}, \qquad w[c; 0, x, y] < \frac{\Delta}{4} \tag{5.35}$$

are valid.

The inequalities (5.22) and (5.35) prove the lemma.

6. Proof of Theorem 3.1. We construct the functions v and u° satisfying the conditions of Theorem 4.1. We select a number c_0 small enough to satisfy the conditions

$$w[c_0; 0, x, y] < \frac{\varepsilon^2(1-p)(\|x\|^2+\|y\|^2)}{8N^2} \tag{6.1}$$

$$\int_0^T \varphi(t)\sigma^2(t)\,dt < \frac{\varepsilon^2(1-p)}{8n^2}$$
$$\varphi(t) = \max\{|\alpha_{ij}(c_0, t)|, |\beta_{ij}(c_0, t)|, \gamma_{ij}(c_0, t)|\} \tag{6.2}$$

Such a number c_0 can be selected in accordance with Lemma 5.1.

We set

$$v[t, x, y] = w[c_0; t, x, y] + \int_t^T (2n)^2\sigma^2(\tau)\varphi(\tau)\,d\tau \tag{6.3}$$

$$u^\circ[t, x, y] = u^*[c_0; t, x, y] \tag{6.4}$$

For such a choice the functions v and u° will satisfy the conditions of Theorem 4.1, and consequently the conditions of Theorem 3.1 as well. Indeed, by virtue of (5.4), (6.1) and (6.2), the function v satisfies the condition

$$v[0, x, y] < (1-p)\varepsilon^2, \qquad v[T, x, y] = \|x(T)\|^2$$

The derivative $(dM\{v\}/dt$; (1.1), (1.6), $u^\circ = u^*)$ differs from the derivative $(d\{v\}/dt$; (5.1), $u^\circ = u^*)$ by the term

$$\frac{1}{2}\sum_{i,j=1}^n \left[\frac{\partial^2 v}{\partial x_i\,\partial x_j} - 2\,\frac{\partial^2 v}{\partial x_i\,\partial y_j} + \frac{\partial^2 v}{\partial y_i\,\partial y_j}\right]\sigma_{ij}$$

and therefore

$$\left(\frac{dM\{v\}}{dt};\ (1.1),\ (1.6),\ u^\circ = u^*\right) = -c_0(u^\circ)^2 - 4n^2\sigma^2(t)\varphi(t) + \\ + \sum_{i,j=1}^n [\alpha_{ij}(c_0, t) - 2\beta_{ij}(c_0, t) + \gamma_{ij}(c_0, t)]\sigma_{ij} < -c_0(u^\circ)^2 < 0 \tag{6.5}$$

Thus, the conditions of Theorem 4.1 are indeed satisfied. This means that the function $u^{\circ} = u^{*}[c_0\ \ t,\ x,\ y]$ satisfies the conditions of Theorem 3.1, which proves this theorem.

We note that the control function u° we have constructed has a minimum property, namely, that the control function $u^{\circ} = u^{*}$ minimizes the functional

$$J^{*} = M\left\{\int_0^T \left[c_0 u^2(t) + 4n^2\sigma^2(t)\varphi(t) - \right.\right.$$

$$\left.\left. - \sum_{i,j=1}^{n} [\alpha_{ij}(c_0, t) - 2\beta_{ij}(c_0, t) + \gamma_{ij}(c_0, t)]\,\sigma_{ij}(t, y(t))\right] dt + \|x(T)\|^2\right\}$$

where

$$\min J^{*} = v\,[0, x, y] \tag{6.6}$$

In conclusion, we note one more fact. The control function u° which solves the tracking problem A is constructed by the following rule: At each instant of time $t = \tau$, for the realized values $x(\tau)$, $y(\tau)$ the control function $u^{\circ}[\tau,\ x(\tau),\ y(\tau)]$ coincides with the control function u^{*} of $[c_0;\ \tau,\ x(\tau),\ y(\tau)]$, which is the solution of the auxiliary optimal control problem $B(c)$ (and which also assures that the point $x(t)$ will be brought into a small neighborhood of the point $x = 0$ at time $t = T$).

The control function u^{*} is an optimal control function for the system (5.1) and minimizes (5.2), where for $t > \tau$ the function $y(t)$ is deterministic and coincides with $M\{y(t)|\,y(\tau)\}$ of the stochastic system. In other words, the control function u° in the stochastic tracking problem is constructed at each instant of time $t = \tau$ at the point $x(\tau)$, $y(\tau)$ in the same way as it would be constructed in an analogous deterministic tracking problem, where the future behavior of the tracked motion $y(t)$ $(t > \tau)$ would coincide with the prediction of the future mathematical expectation of the tracked motion. Naturally, the control function $u^{\circ} = u^{*}$ must be chosen to be sufficiently strong (the number $c = c_0$ in the auxiliary problem $B(c)$ must be sufficiently small).

An example of the problem considered in the present article may be found in the problem of making a rotating body which is controlled (by changes in moment) agree within a time T (in angle and speed of rotation) with a rotating body subject to random pulse moments.

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